The analogue of Wilczynski's projective frame in Lie sphere geometry: Lie-applicable surfaces and commuting Schrödinger operators with magnetic fields.

Ferapontov E.V.
Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom
e-mail: E.V.Ferapontov@lboro.ac.uk
and
Centre for Nonlinear Studies
Landau Institute of Theoretical Physics
Academy of Science of Russia, Kosygina 2
117940 Moscow, GSP-1, Russia
e-mail: fer@landau.ac.ru

Abstract

We propose a twistor construction of surfaces in Lie sphere geometry based on the linear system which copies equations of Wilczynski's projective frame. In the particular case of Lie-applicable surfaces this linear system describes joint eigenfunctions of a pair of commuting Schrödinger operators with magnetic terms.

Keywords: Lie sphere geometry, Wilczynski's frame, twistor methods, Schrödinger operators with magnetic terms.

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1 Introduction

Let M^2 be a surface in E^3 parametrized by coordinates R^1, R^2 of curvature lines, with the radius-vector $\mathbf{r}$ and the unit normal $\mathbf{n}$ satisfying the Weingarten equations

$$\begin{aligned} \partial_1 \mathbf{r} &= w^1 \partial_1 \mathbf{n} \\ \partial_2 \mathbf{r} &= w^2 \partial_2 \mathbf{n}, \end{aligned} \tag{1}$$

where w^1 and w^2 are the radii of principal curvature, $\partial_1 = \partial/\partial R^1$, $\partial_2 = \partial/\partial R^2$. Let us recall the construction of the Lie sphere map [11]. With any sphere $S(R, \mathbf{c})$ having radius R and center $\mathbf{c} = (c^1, c^2, c^3)$ this map associates the 6-vector $\{y_0, y_1, y_2, y_3, y_4, y_5\}$ with hexaspherical coordinates

$$\left\{ \frac{1 + \mathbf{c}^2 - R^2}{2}, \ \frac{1 - \mathbf{c}^2 + R^2}{2}, \ \mathbf{c}, \ R \right\},$$

which obey the relation

$$-y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_5^2 = 0. \tag{2}$$

This equation defines the so-called Lie quadric. Thus, with any sphere $S(R, \mathbf{c})$ in E^3 we associate a point on the Lie quadric (2). The Lie sphere map linearises the action of the Lie sphere group which is a group of contact transformations in E^3 generated by conformal transformations and normal shifts. In hexaspherical coordinates, the action of the Lie sphere group coincides with the linear action of $SO(4,2)$ which preserves the Lie quadric (2). The reader may consult [11], [1], [3], [14], [15] for further properties of this construction. Applying Lie sphere map to the curvature spheres $S(w^1, \mathbf{r} - w^1\mathbf{n})$ and $S(w^2, \mathbf{r} - w^2\mathbf{n})$ of the surface M^2, we obtain a pair of two-dimensional submanifolds

$$\mathbf{U} = \left\{ \frac{1 + \mathbf{r}^2 - 2w^1(\mathbf{r}, \mathbf{n})}{2}, \ \frac{1 - \mathbf{r}^2 + 2w^1(\mathbf{r}, \mathbf{n})}{2}, \ \mathbf{r} - w^1\mathbf{n}, \ w^1 \right\}$$

and

$$\mathbf{V} = \left\{ \frac{1 + \mathbf{r}^2 - 2w^2(\mathbf{r}, \mathbf{n})}{2}, \ \frac{1 - \mathbf{r}^2 + 2w^2(\mathbf{r}, \mathbf{n})}{2}, \ \mathbf{r} - w^2\mathbf{n}, \ w^2 \right\}$$

of the Lie quadric. Blaschke's approach [1] to the Lie sphere geometry of surfaces in 3-space is based on the following two simple facts:

(a) By construction, vectors $\mathbf{U}$ and $\mathbf{V}$ have zero norm

$$(\mathbf{U}, \ \mathbf{U}) = (\mathbf{V}, \ \mathbf{V}) = 0$$

in the scalar product defined by (2).

(b) The triple $\mathbf{U}, \ \partial_2\mathbf{U}, \ \partial_2^2\mathbf{U}$ is orthogonal to the triple $\mathbf{V}, \ \partial_1\mathbf{V}, \ \partial_1^2\mathbf{V}$.

Using appropriate linear combinations among these triples, one can construct an invariant 6-frame canonically associated with a surface M^2 (Appendix B, see also [1], [4]). Conversely, given a pair of 6-vectors $\mathbf{U}$ and $\mathbf{V}$ satisfying (a) and (b), the surface can be reconstructed uniquely as the envelope of the corresponding family of curvature spheres.

Our construction of vectors $\mathbf{U}$ and $\mathbf{V}$ which satisfy the above properties was borrowed from projective differential geometry. In 1907, Wilczynski [17] proposed the approach to surfaces in projective space based on a linear system

$$\begin{aligned} \mathbf{r}_{xx} &= \beta\ \mathbf{r}_y + \tfrac{1}{2}(V - \beta_y)\ \mathbf{r} \\ \mathbf{r}_{yy} &= \gamma\ \mathbf{r}_x + \tfrac{1}{2}(W - \gamma_x)\ \mathbf{r} \end{aligned} \tag{3}$$

where β, γ, V, W are real functions of x and y. Cross-differentiating (3) and assuming $\mathbf{r}, \mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_{xy}$ to be independent, we arrive at the compatibility conditions

$$\begin{gathered} \beta_{yyy} - 2\beta_y W - \beta W_y = \gamma_{xxx} - 2\gamma_x V - \gamma V_x \\ W_x = 2\gamma\beta_y + \beta\gamma_y \\ V_y = 2\beta\gamma_x + \gamma\beta_x. \end{gathered} \tag{4}$$

For any β, γ, V, W satisfying (4) the linear system (3) has rank 4, so that $\mathbf{r} = (r^0 : r^1 : r^2 : r^3)$ can be viewed as a 4-component real vector of homogeneous coordinates of a surface M^2 in a projective space $\mathbf{RP}^3$. The compatibility conditions (4) can be interpreted as the 'Gauss-Codazzi' equations in projective differential geometry. The independent variables x, y play the role of asymptotic coordinates on the surface M^2. Introducing the vectors

$$\mathcal{U} = \mathbf{r} \wedge \mathbf{r}_x \quad \text{and} \quad \mathcal{V} = \mathbf{r} \wedge \mathbf{r}_y$$

in $\Lambda^2(R^4)$, one readily verifies that
(a) vectors $\mathcal{U}$ and $\mathcal{V}$ have zero norm

$$(\mathcal{U},\ \mathcal{U}) = (\mathcal{V},\ \mathcal{V}) = 0$$

in the natural scalar product in $\Lambda^2(R^4)$ defined by the Plücker formula;
(b) the triple $\mathcal{U},\ \mathcal{U}_y,\ \mathcal{U}_{yy}$ is orthogonal to the triple $\mathcal{V},\ \mathcal{V}_x,\ \mathcal{V}_{xx}$.
The passage from a projective surface to a pair of submanifolds $\mathcal{U}$ and $\mathcal{V}$ in the Plücker quadric in $\Lambda^2(R^4)$ plays an important role in projective differential geometry. We refer to [17], [2], [10], [6], [5], [16]) for a further discussion. Some more details on Wilczynski's approach are included in Appendix A.

The main observation of this paper is that a similar approach, based on the linear system

$$\begin{aligned} \partial_1^2\boldsymbol{\psi} &= -ip\ \partial_2\boldsymbol{\psi} + \tfrac{1}{2}(V + i\partial_2 p)\ \boldsymbol{\psi} \\ \partial_2^2\boldsymbol{\psi} &= iq\ \partial_1\boldsymbol{\psi} + \tfrac{1}{2}(W - i\partial_1 q)\ \boldsymbol{\psi}, \end{aligned} \tag{5}$$

applies to Lie sphere geometry. Here $\partial_1 = \partial/\partial R^1$ and $\partial_2 = \partial/\partial R^2$ denote partial derivatives with respect to the independent variables $R^1,\ R^2$, and p, q, V, W are real potentials. This system is a complex analogue of the linear system (3) (indeed, the complex transformation $\beta = -ip,\ \gamma = iq$ identifies both systems). Again, cross-differentiation produces

the compatibility conditions

$$\partial_2^3 p - 2W\partial_2 p - p\partial_2 W + \partial_1^3 q - 2V\partial_1 q - q\partial_1 V = 0$$

$$\partial_1 W = 2q\partial_2 p + p\partial_2 q \tag{6}$$

$$\partial_2 V = 2p\partial_1 q + q\partial_1 p.$$

For any fixed p, q, V, W satisfying (6), the linear system (5) is compatible and its solution space has complex dimension 4, so that we can view $\boldsymbol{\psi}$ as an element of the twistor space $\mathbf{CP}^3$. In what follows, equations (6) will be identified with the 'Gauss-Codazzi' equations in Lie sphere geometry, while the independent variables R^1, R^2 will play the role of curvature line coordinates. An important property of linear system (5) is the existence of the invariant pseudo-Hermitian scalar product $(\ ,\ )$ of the signature $(2,2)$ such that

$$(\boldsymbol{\psi},\ \partial_1\partial_2\boldsymbol{\psi}) = (\partial_1\partial_2\boldsymbol{\psi},\ \boldsymbol{\psi}) = -1, \qquad (\partial_1\boldsymbol{\psi},\ \partial_2\boldsymbol{\psi}) = (\partial_2\boldsymbol{\psi},\ \partial_1\boldsymbol{\psi}) = 1$$

$$(\partial_1\partial_2\boldsymbol{\psi},\ \partial_1\partial_2\boldsymbol{\psi}) = -pq \tag{7}$$

(all other scalar products being zero). Fixing $\boldsymbol{\psi} \in \mathbf{C}^4$ satisfying both (5) and (7), we define two real vectors $\mathbf{U}$ and $\mathbf{V}$ in $\Lambda^2(\mathbf{C}^4)$ as

$$\mathbf{U} = -2\ Im(\boldsymbol{\psi} \wedge \partial_1\boldsymbol{\psi}), \quad \mathbf{V} = 2\ Re(\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}).$$

Introducing in $\Lambda^2(\mathbf{C}^4)$ a pseudo-Hermitian scalar product $(\ ,\ )$ induced by (7) (the wedge product $\wedge$ and the scalar product $(\ ,\ )$ in $\Lambda^2(\mathbf{C}^4)$ are explicitly defined in Appendix C), we show that the restriction of $(\ ,\ )$ to the 6-dimensional invariant real subspace in $\Lambda^2(\mathbf{C}^4)$ spanned by $\mathbf{U},\ \partial_2\mathbf{U},\ \partial_2^2\mathbf{U}$ and $\mathbf{V},\ \partial_1\mathbf{V},\ \partial_1^2\mathbf{V}$ is a real scalar product of the signature $(4,2)$. Moreover,
(a) vectors $\mathbf{U}$ and $\mathbf{V}$ have zero norm

$$(\mathbf{U},\ \mathbf{U}) = (\mathbf{V},\ \mathbf{V}) = 0;$$

(b) the triple $\mathbf{U},\ \partial_2\mathbf{U},\ \partial_2^2\mathbf{U}$ is orthogonal to the triple $\mathbf{V},\ \partial_1\mathbf{V},\ \partial_1^2\mathbf{V}$.
Thus, $\mathbf{U}$ and $\mathbf{V}$ are hexaspherical coordinates of curvature spheres of a surface M^2 parametrized by curvature line coordinates R^1, R^2. In fact, our construction is based on the isomorphism $SU(2,2)/\pm 1 \to SO(4,2)$ which is a basis of twistor theory [13], [9], [12].

The main motivation for the construction described above comes from the study of Lie-applicable surfaces. We recall that two surfaces are called Lie-applicable if, being non-equivalent under Lie sphere transformations, they have the same coefficients p and q in the appropriate curvature line parametrization R^1, R^2 [1]. Analytically, Lie-applicable surfaces are described by equations (6) which, for given p and q, are not uniquely solvable for V and W. Examples presented in section 3 demonstrate that for some particularly interesting classes of Lie-applicable surfaces, equations (5) describe joint eigenfunctions of a pair of commuting Schrödinger operators with magnetic fields. These examples include nonsingular doubly periodic operators on a two-dimensional torus and the Dirac monopole

on a two-dimensional sphere [18]. It should be emphasized that this important structure is not visible in the standard approach based on Blaschke's 6-frame, becoming apparent only after applying the twistor construction.

In section 4 we give a separate treatment of canal surfaces (which are characterized by a condition $q = 0$ (or $p = 0$)), since the construction of the Lie sphere frame adopted in section 2 does not automatically carry over to this case. As an example, we explicitly construct the surfaces of revolution for which equations (5) reduce to eigenfunction equations of the Schrödinger operator in a homogeneous magnetic field. The case of the first nontrivial Landau level is worked out in detail.

2 Analog of Wilczynski's projective frame in Lie sphere geometry

In this section we propose an approach to surfaces in Lie sphere geometry based on the linear system (5) satisfying the compatibility conditions (6). Our further constructions will follow those from projective differential geometry — see Appendix A. Notice first that system (5) is covariant under transformations of the form

$$(R^1)^* = f(R^1), \qquad (R^2)^* = g(R^2), \qquad \boldsymbol{\psi}^* = \sqrt{f'(R^1)g'(R^2)}\;\boldsymbol{\psi} \tag{8}$$

which act on the potentials p, q, V, W as follows

$$\begin{aligned} p^* &= pg'/(f')^2, \qquad V^*(f')^2 = V + S(f) \\ q^* &= qf'/(g')^2, \qquad W^*(g')^2 = W + S(g). \end{aligned} \tag{9}$$

Here $S(\,\cdot\,)$ is the Schwarzian derivative

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2,$$

so that V and W can be interpreted as projective connections. Transformation formulae (8), (9) imply that the symmetric 2-form

$$-pq\; dR^1 dR^2 \tag{10}$$

and the conformal class of the cubic form

$$p(dR^1)^3 - q(dR^2)^3 \tag{11}$$

are invariant. In what follows they will play the roles of the Lie-invariant metric and the Lie-invariant cubic form, respectively. In this section we assume both p and q to be nonzero. Let us introduce the four vectors

$$\begin{aligned} &\boldsymbol{\psi}, \quad \boldsymbol{\psi}_1 = \partial_1\boldsymbol{\psi} - \tfrac{1}{2}\tfrac{\partial_1 q}{q}\boldsymbol{\psi}, \quad \boldsymbol{\psi}_2 = \partial_2\boldsymbol{\psi} - \tfrac{1}{2}\tfrac{\partial_2 p}{p}\boldsymbol{\psi}, \\ &\boldsymbol{\eta} = \partial_1\partial_2\boldsymbol{\psi} - \tfrac{1}{2}\tfrac{\partial_1 q}{q}\partial_2\boldsymbol{\psi} - \tfrac{1}{2}\tfrac{\partial_2 p}{p}\partial_1\boldsymbol{\psi} + \left(\tfrac{1}{4}\tfrac{\partial_2 p\,\partial_1 q}{pq} - \tfrac{1}{2}pq\right)\boldsymbol{\psi} \end{aligned} \tag{12}$$

which are straightforward analogues of vertices of Wilczynski's moving tetrahedral — see formulae (43) in Appendix A. Notice that under transformations (8) vectors (12) acquire nonzero multiples which do not change them as points in the complex projective space. Using (5) and (12), we readily derive for $\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\eta}$ the linear system

$$
\begin{aligned}
\partial_1 \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\frac{\partial_1 q}{q} & 1 & 0 & 0 \\ \frac{1}{2}b & -\frac{1}{2}\frac{\partial_1 q}{q} & -ip & 0 \\ \frac{1}{2}k & 0 & \frac{1}{2}\frac{\partial_1 q}{q} & 1 \\ -\frac{i}{2}pa & \frac{1}{2}k & \frac{1}{2}b & -\frac{1}{2}\frac{\partial_1 q}{q} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} \\
\partial_2 \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\frac{\partial_2 p}{p} & 0 & 1 & 0 \\ \frac{1}{2}l & \frac{1}{2}\frac{\partial_2 p}{p} & 0 & 1 \\ \frac{1}{2}a & iq & -\frac{1}{2}\frac{\partial_2 p}{p} & 0 \\ \frac{i}{2}qb & \frac{1}{2}a & \frac{1}{2}l & -\frac{1}{2}\frac{\partial_2 p}{p} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix}
\end{aligned} \tag{13}
$$

where we introduced the notation

$$
k = pq - \partial_1\partial_2(\ln p), \qquad l = pq - \partial_1\partial_2(\ln q)
$$

$$
a = W - \partial_2^2(\ln p) - \tfrac{1}{2}(\partial_2 \ln p)^2, \qquad b = V - \partial_1^2(\ln q) - \tfrac{1}{2}(\partial_1 \ln q)^2.
$$

The compatibility conditions of equations (13) imply

$$
\begin{gathered}
\partial_1\partial_2 \ln p = pq - k, \qquad \partial_1\partial_2 \ln q = pq - l \\
\partial_1 a = \partial_2 k + \tfrac{\partial_2 p}{p}\, k, \qquad \partial_2 b = \partial_1 l + \tfrac{\partial_1 q}{q}\, l \\
p\,\partial_2 a + 2\,a\,\partial_2 p + q\,\partial_1 b + 2\,b\,\partial_1 q = 0
\end{gathered} \tag{14}
$$

which is just an equivalent form of the 'Gauss-Codazzi' equations (6). An important property of system (13) is the existence of the quadratic integral

$$
-\boldsymbol{\psi}\bar{\boldsymbol{\eta}} + \boldsymbol{\psi}_1\bar{\boldsymbol{\psi}}_2 + \boldsymbol{\psi}_2\bar{\boldsymbol{\psi}}_1 - \boldsymbol{\eta}\bar{\boldsymbol{\psi}} \tag{15}
$$

which defines an invariant pseudo-Hermitian scalar product of the signature $(2, 2)$ on the space of solutions of system (13). Using (12), this integral can be rewritten in the form

$$
-\boldsymbol{\psi}\ \partial_1\partial_2\bar{\boldsymbol{\psi}} + \partial_1\boldsymbol{\psi}\ \partial_2\bar{\boldsymbol{\psi}} + \partial_2\boldsymbol{\psi}\ \partial_1\bar{\boldsymbol{\psi}} - \bar{\boldsymbol{\psi}}\ \partial_1\partial_2\boldsymbol{\psi} + pq\ \boldsymbol{\psi}\bar{\boldsymbol{\psi}}. \tag{16}
$$

The existence of the invariant pseudo-Hermitian scalar product (15) allows to choose a basis of solutions of (13) such that

$$
(\boldsymbol{\psi}_1,\ \boldsymbol{\psi}_2) = (\boldsymbol{\psi}_2,\ \boldsymbol{\psi}_1) = 1, \qquad (\boldsymbol{\psi},\ \boldsymbol{\eta}) = (\boldsymbol{\eta},\ \boldsymbol{\psi}) = -1, \tag{17}
$$

all other scalar products being zero. Notice that formulae (17) are equivalent to (7). Here $(\ ,\ )$ denotes the pseudo-Hermitian scalar product in $\mathbf{C}^4$

$$
(\mathbf{a}, \mathbf{b}) = -a^0\bar{b}^3 + a^1\bar{b}^2 + a^2\bar{b}^1 - a^3\bar{b}^0
$$

of the signature (2, 2). Equations (13) also imply that the determinant $\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}$ is invariant:

$$\partial_1(\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}) = \partial_2(\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}) = 0$$

(indeed, both matrices in (13) are traceless). Thus, besides (17), we can impose the additional constraint

$$\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta} = 1. \tag{18}$$

From now on, we fix a null-tetrad $\boldsymbol{\psi},\ \boldsymbol{\psi}_1,\ \boldsymbol{\psi}_2,\ \boldsymbol{\eta}$ satisfying both (17) and (18). Notice that this basis $\boldsymbol{\psi},\ \boldsymbol{\psi}_1,\ \boldsymbol{\psi}_2,\ \boldsymbol{\eta}$ described above is defined up to a natural linear action of the group $SU(2,2)$ which preserves both (17) and (18). Introducing the basis in $\Lambda^2\mathbf{C}^4$ as follows (notice the analogy with self-dual and anti-self-dual forms in twistor theory)

$$\mathcal{U} = i\ \boldsymbol{\psi}\wedge\boldsymbol{\psi}_1, \quad \mathcal{V} = \boldsymbol{\psi}\wedge\boldsymbol{\psi}_2,$$

$$\mathcal{A} = i\ \boldsymbol{\psi}_2\wedge\boldsymbol{\psi}_1 + i\ \boldsymbol{\psi}\wedge\boldsymbol{\eta}, \quad \mathcal{B} = \boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2 + \boldsymbol{\psi}\wedge\boldsymbol{\eta},$$

$$\mathcal{P} = 2i\,\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}, \quad \mathcal{Q} = 2\,\boldsymbol{\psi}_1\wedge\boldsymbol{\eta},$$

we arrive at the equations

$$\begin{gathered}
\partial_1\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & p & 0 & 0\\ k & 0 & 0 & 0 & 0 & 0\\ 0 & k & 0 & -pa & 0 & 0\\ 0 & 0 & 0 & \frac{\partial_1 q}{q} & 1 & 0\\ 0 & 0 & 0 & b & 0 & 1\\ pa & 0 & -p & 0 & b & -\frac{\partial_1 q}{q}\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}\\
\partial_2\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix} \frac{\partial_2 p}{p} & 1 & 0 & 0 & 0 & 0\\ a & 0 & 1 & 0 & 0 & 0\\ 0 & a & -\frac{\partial_2 p}{p} & qb & 0 & -q\\ q & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & l & 0 & 0\\ -qb & 0 & 0 & 0 & l & 0\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}
\end{gathered} \tag{19}$$

which identically coincide with equations of the Lie sphere frame — see Appendix B. Let us define the wedge product $\wedge$ as in Appendix C and introduce the pseudo-Hermitian scalar product (,) in $\Lambda^2\mathbf{C}^4$

$$(\mathbf{a}\wedge\mathbf{b},\ \mathbf{A}\wedge\mathbf{B}) = \frac{1}{2}((\mathbf{a},\mathbf{B})(\mathbf{b},\mathbf{A}) - (\mathbf{a},\mathbf{A})(\mathbf{b},\mathbf{B}))$$

(we hope that the same notation (,) for pseudo-Hermitian scalar products in $\mathbf{C}^4$ and $\Lambda^2\mathbf{C}^4$ will not lead to a confusion). A direct computation shows that the only nonzero products among the vectors $\mathcal{U},\mathcal{A},\mathcal{P},\mathcal{V},\mathcal{B},\mathcal{Q}$ are

$$(\mathcal{U},\ \mathcal{P}) = (\mathcal{P},\ \mathcal{U}) = (\mathcal{V},\ \mathcal{Q}) = (\mathcal{Q},\ \mathcal{V}) = -1, \quad (\mathcal{A},\ \mathcal{A}) = (\mathcal{B},\ \mathcal{B}) = 1. \tag{20}$$

This invariant pseudo-Hermitian scalar product corresponds to the quadratic integral

$$\mathcal{A}\bar{\mathcal{A}} + \mathcal{B}\bar{\mathcal{B}} - \mathcal{U}\bar{\mathcal{P}} - \mathcal{P}\bar{\mathcal{U}} - \mathcal{V}\bar{\mathcal{Q}} - \mathcal{Q}\bar{\mathcal{V}}$$

of system (19). Similarly, we can define the complex scalar product $\{\ ,\ \}$ in $\Lambda^2(\mathbf{C}^4)$ (see Appendix C):

$$\{\mathbf{a} \wedge \mathbf{b},\ \mathbf{A} \wedge \mathbf{B}\} = \frac{1}{2}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{A} \wedge \mathbf{B})$$

A direct computation shows that the only nonzero products among the vectors $\mathcal{U}, \mathcal{A}, \mathcal{P}, \mathcal{V}, \mathcal{B}, \mathcal{Q}$ are

$$\{\mathcal{U},\ \mathcal{P}\} = \{\mathcal{P},\ \mathcal{U}\} = \{\mathcal{V},\ \mathcal{Q}\} = \{\mathcal{Q},\ \mathcal{V}\} = -1, \quad \{\mathcal{A},\ \mathcal{A}\} = \{\mathcal{B},\ \mathcal{B}\} = 1. \tag{21}$$

This invariant complex scalar product corresponds to the quadratic integral

$$\mathcal{A}^2 + \mathcal{B}^2 - 2\mathcal{U}\mathcal{P} - 2\mathcal{V}\mathcal{Q}$$

of system (19). Notice the important relation between the pseudo-Hermitian scalar product $(\ ,\ )$ and the complex scalar product $\{\ ,\ \}$ in $\Lambda^2(\mathbf{C}^4)$:

$$(\xi,\ \psi) = \{\xi,\ \bar{\psi}\} \tag{22}$$

for any ξ, ψ in $\Lambda^2(\mathbf{C}^4)$ (see Appendix 3). Let us finally introduce in $\Lambda^2(\mathbf{C}^4)$ the real vectors

$$\mathbf{U} = \mathcal{U} + \bar{\mathcal{U}} = i\ \boldsymbol{\psi} \wedge \partial_1 \boldsymbol{\psi} - i\ \overline{\boldsymbol{\psi} \wedge \partial_1 \boldsymbol{\psi}}, \qquad \mathbf{V} = \mathcal{V} + \bar{\mathcal{V}} = \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi} + \overline{\boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}}.$$

Theorem 1 *Vectors* $\mathbf{U}$ *and* $\mathbf{V}$ *have zero norm:*

$$(\mathbf{U},\ \mathbf{U}) = (\mathbf{V},\ \mathbf{V}) = 0.$$

Moreover, the triple $\mathbf{U},\ \partial_2\mathbf{U},\ \partial_2^2\mathbf{U}$ *is orthogonal to the triple* $\mathbf{V},\ \partial_1\mathbf{V},\ \partial_1^2\mathbf{V}$. *Hence,* $\mathbf{U}$ *and* $\mathbf{V}$ *are curvature spheres of a surface.*

Proof:

It readily follows from (19) that the triple $\mathbf{U},\ \partial_2\mathbf{U},\ \partial_2^2\mathbf{U}$ is equivalent to $\mathcal{U} + \bar{\mathcal{U}},\ \mathcal{V} + \bar{\mathcal{V}},\ \mathcal{P} + \bar{\mathcal{P}}$. Similarly, the triple $\mathbf{V},\ \partial_1\mathbf{V},\ \partial_1^2\mathbf{V}$ is equivalent to $\mathcal{V} + \bar{\mathcal{V}},\ \mathcal{B} + \bar{\mathcal{B}},\ \mathcal{Q} + \bar{\mathcal{Q}}$. Conditions $(\mathbf{U},\ \mathbf{U}) = (\mathbf{V},\ \mathbf{V}) = 0$ and the orthogonality of both triples follow by virtue of (20), (21) and (22). Let us show, for instance, that $(\mathbf{U},\ \mathbf{U}) = 0$:

$$(\mathbf{U},\ \mathbf{U}) = (\mathcal{U} + \bar{\mathcal{U}},\ \mathcal{U} + \bar{\mathcal{U}}) = (\mathcal{U},\ \mathcal{U}) + (\mathcal{U},\ \bar{\mathcal{U}}) + (\bar{\mathcal{U}},\ \mathcal{U}) + (\bar{\mathcal{U}},\ \bar{\mathcal{U}}) =$$

$$(\mathcal{U},\ \mathcal{U}) + \{\mathcal{U},\ \mathcal{U}\} + \overline{(\mathcal{U},\ \bar{\mathcal{U}})} + \{\bar{\mathcal{U}},\ \mathcal{U}\} = (\mathcal{U},\ \mathcal{U}) + \{\mathcal{U},\ \mathcal{U}\} + \overline{\{\mathcal{U},\ \mathcal{U}\}} + \{\mathcal{U},\ \bar{\mathcal{U}}\} =$$

$$(\mathcal{U},\ \mathcal{U}) + \{\mathcal{U},\ \mathcal{U}\} + \overline{\{\mathcal{U},\ \mathcal{U}\}} + (\mathcal{U},\ \mathcal{U}),$$

which is zero by virtue of (20) and (21).

Remark 1. It is straightforward to show that any surface can be obtained (locally) by a construction described above.

Remark 2. In view of (7) the surface $\boldsymbol{\psi}(R^1, R^2) \in \mathbf{CP}^3$ defines a Legendre submanifold of the quadric $(\boldsymbol{\psi},\ \boldsymbol{\psi}) = 0$ equipped with a real contact form $i(\boldsymbol{\psi},\ d\boldsymbol{\psi})$. I would like to thank L. Mason for clarifying this point.

3 Surfaces possessing 3-parameter families of Lie deformations and commuting Schrödinger operators with magnetic fields

In contrast with the Euclidean geometry, where a surface is uniquely determined by its first and second fundamental forms, there exist examples of surfaces in Lie sphere geometry which are not uniquely specified by the Lie-invariant metric

$$-pq\ dR^1 dR^2$$

and the conformal class of the cubic form

$$p\,(dR^1)^3 - q\,(dR^2)^3.$$

Such surfaces are called Lie-applicable (Lie-deformable). In this section we consider examples of surfaces possessing 3-parameter families of Lie deformations. A calculation similar to the one done by Finikov in [8] shows that these surfaces are characterized by the constraints

$$\partial_1\partial_2 \ln p = c\ pq, \quad \partial_1\partial_2 \ln q = c\ pq, \tag{23}$$

where c is a constant. There are different cases to distinguish depending on the value of c. Here we discuss the two simplest cases $c = 0$ and $c = 1$ (for $c \neq 0, 1$ the formulae become more complicated).

Case c=0 Utilizing transformations (8) and (9), we can represent p and q in the form

$$p = \psi_1'(R^1), \quad q = -\psi_2'(R^2),$$

implying, after the substitution into (6) and elementary integration, the following expressions for V and W

$$V = \epsilon_1 + \epsilon_0\psi_1 - \psi_2\psi_1'' - \frac{1}{2}\rho_2\psi_1^2, \quad W = \epsilon_2 + \epsilon_0\psi_2 - \psi_1\psi_2'' - \frac{1}{2}\rho_1\psi_2^2.$$

Moreover, ψ_1 and ψ_2 satisfy the ODE's

$$\psi_1'' = \alpha\psi_1^2 + \rho_1\psi_1 + s_1, \quad \psi_2'' = \alpha\psi_2^2 + \rho_2\psi_2 + s_2$$

implying that ψ_1 and ψ_2 are elliptic functions. Here $\epsilon_0, \epsilon_1, \epsilon_2, \alpha, \rho_1, \rho_2, s_1, s_2$ are arbitrary constants (if α is nonzero one can always reduce ρ_1 and ρ_2 to zero by adding constants to ψ_1, ψ_2). Notice that for given p and q the corresponding V and W are determined up to three arbitrary constants $\epsilon_0, \epsilon_1, \epsilon_2$, which are thus responsible for Lie deformations. It is important to emphasize the linear dependence of V and W on the deformation parameters. This readily follows from (6), indeed, for given p and q these equations are linear in V and W. The corresponding system (5) takes the form

$$\begin{aligned}
\partial_1^2\boldsymbol{\psi} &= -i\psi_1'\ \partial_2\boldsymbol{\psi} + \tfrac{1}{2}(\epsilon_1 + \epsilon_0\psi_1 - \psi_2\psi_1'' - \tfrac{1}{2}\rho_2\psi_1^2)\ \boldsymbol{\psi},\\
\partial_2^2\boldsymbol{\psi} &= -i\psi_2'\ \partial_1\boldsymbol{\psi} + \tfrac{1}{2}(\epsilon_2 + \epsilon_0\psi_2 - \psi_1\psi_2'' - \tfrac{1}{2}\rho_1\psi_2^2)\ \boldsymbol{\psi},
\end{aligned} \tag{24}$$

which, upon the addition and subtraction, readily rewrites in the form

$$H\boldsymbol{\psi} = \lambda\ \boldsymbol{\psi}, \quad F\boldsymbol{\psi} = \mu\ \boldsymbol{\psi}.$$

Here H and F are commuting Schrödinger operators with magnetic terms

$$H = \left(\partial_1 + \frac{i}{2}\psi_2'\right)^2 + \left(\partial_2 + \frac{i}{2}\psi_1'\right)^2 + V_H,$$

$$F = \left(\partial_1 - \frac{i}{2}\psi_2'\right)^2 - \left(\partial_2 - \frac{i}{2}\psi_1'\right)^2 + V_F,$$

$\lambda = \frac{\epsilon_1+\epsilon_2}{2}$ and $\mu = \frac{\epsilon_1-\epsilon_2}{2}$ are the eigenvalues, and the potentials V_H and V_F are given by

$$V_H = \frac{1}{4}(2\psi_2\psi_1'' + 2\psi_1\psi_2'' + \rho_2\psi_1^2 + \rho_1\psi_2^2 + (\psi_2')^2 + (\psi_1')^2 - 2\epsilon_0(\psi_1 + \psi_2)),$$

$$V_F = \frac{1}{4}(2\psi_2\psi_1'' - 2\psi_1\psi_2'' + \rho_2\psi_1^2 - \rho_1\psi_2^2 + (\psi_2')^2 - (\psi_1')^2 - 2\epsilon_0(\psi_1 - \psi_2)).$$

For generic values of constants, operators H and F will be non-singular and doubly periodic. The spectral theory of these operators will be discussed elsewhere.

Case c=1 Here

$$\partial_1\partial_2 \ln p = pq, \quad \partial_1\partial_2 \ln q = pq,$$

implying

$$p = \frac{1}{R^2 - R^1}\sqrt{\frac{f_2}{f_1}}, \quad q = \frac{1}{R^1 - R^2}\sqrt{\frac{f_1}{f_2}},$$

where f_1 and f_2 are functions of R^1 and R^2, respectively. The corresponding V and W are given by

$$V = \partial_1^2(\ln q) + \tfrac{1}{2}(\partial_1 q/q)^2 - \tfrac{\epsilon_0+\epsilon_1 R^1+\epsilon_2 (R^1)^2}{f_1},$$

$$W = \partial_2^2(\ln p) + \tfrac{1}{2}(\partial_2 p/p)^2 + \tfrac{\epsilon_0+\epsilon_1 R^2+\epsilon_2 (R^2)^2}{f_2},$$

where the constants $\epsilon_0, \epsilon_1, \epsilon_2$ are responsible for Lie deformations. These surfaces are known to have both families of curvature lines in linear complexes [1]. Introducing the rescaled vector $\mathbf{R}$ by the formula

$$\boldsymbol{\psi} = (f_1 f_2)^{1/4}\mathbf{R},$$

we readily rewrite equations (5) in the equivalent form

$$\begin{gathered} f_1\ \partial_1^2\mathbf{R} + \tfrac{1}{2}f_1'\ \partial_1\mathbf{R} + i\tfrac{\sqrt{f_1 f_2}}{R^2-R^1}\ \partial_2\mathbf{R} = \\ \left(\tfrac{3f_1}{4(R^2-R^1)^2} + \tfrac{f_1'}{4(R^2-R^1)} - \tfrac{i}{2}\tfrac{\sqrt{f_1 f_2}}{(R^2-R^1)^2} - \tfrac{\epsilon_0+\epsilon_1 R^1+\epsilon_2 (R^1)^2}{2}\right)\mathbf{R}, \\ f_2\ \partial_2^2\mathbf{R} + \tfrac{1}{2}f_2'\ \partial_2\mathbf{R} + i\tfrac{\sqrt{f_1 f_2}}{R^2-R^1}\ \partial_1\mathbf{R} = \\ \left(\tfrac{3f_2}{4(R^2-R^1)^2} - \tfrac{f_2'}{4(R^2-R^1)} + \tfrac{i}{2}\tfrac{\sqrt{f_1 f_2}}{(R^2-R^1)^2} + \tfrac{\epsilon_0+\epsilon_1 R^2+\epsilon_2 (R^2)^2}{2}\right)\mathbf{R}, \end{gathered} \tag{25}$$

where f_1' and f_2' denote the derivatives of $f_1(R^1)$ and $f_2(R^2)$, respectively. Solving for $\frac{\epsilon_1}{2}\mathbf{R}$ and $\frac{\epsilon_0}{2}\mathbf{R}$, we arrive at the eigenfunction equations

$$H\mathbf{R} + \frac{\epsilon_1}{2}\mathbf{R} = 0, \qquad F\mathbf{R} + \frac{\epsilon_0}{2}\mathbf{R} = 0$$

where the operator H is of the form

$$\sqrt{g^{11}g^{22}}\,(i\partial_x + A)\,\frac{g^{11}}{\sqrt{g^{11}g^{22}}}\,(i\partial_x + A) + \sqrt{g^{11}g^{22}}\,(i\partial_y + B)\,\frac{g^{22}}{\sqrt{g^{11}g^{22}}}\,(i\partial_y + B) + h. \tag{26}$$

Here g^{11} and g^{22} are the components of a diagonal metric of Stäckel type

$$g^{11} = \frac{f_1}{R^2 - R^1}, \quad g^{22} = \frac{f_2}{R^2 - R^1},$$

A and B are the components of the magnetic vector potential

$$A = -\frac{1}{2(R^2 - R^1)}\sqrt{\frac{f_2}{f_1}}, \quad B = -\frac{1}{2(R^2 - R^1)}\sqrt{\frac{f_1}{f_2}},$$

and h is the scalar potential

$$h = \frac{f_1' - f_2'}{4(R^2 - R^1)^2} + \frac{f_1 + f_2}{2(R^2 - R^1)^3} + \frac{\epsilon_2}{2}(R^1 + R^2).$$

Geometrically, H represents the Laplace-Beltrami operator corresponding to the metric g^{11}, g^{22} in the magnetic potential A, B and the scalar potential h. Notice that the scalar potential h can be represented in a simple coordinate-free form

$$h = K + \frac{\epsilon_2}{2}(R^1 + R^2)$$

where K is the Gaussian curvature of the metric $g^{11},\ g^{22}$. The second term $R^1 + R^2$ is nothing but the trace of the Killing tensor of the Stäckel metric $g^{11},\ g^{22}$, and hence also makes an invariant sense. Computation of the magnetic field implies

$$(\partial_1 B - \partial_2 A)\, dR^1 \wedge dR^2 = -\left(\frac{f_1' - f_2'}{4(R^2 - R^1)^2} + \frac{f_1 + f_2}{2(R^2 - R^1)^3}\right)\frac{R^2 - R^1}{\sqrt{f_1 f_2}}\, dR^1 \wedge dR^2 = -K\, d\sigma$$

where

$$d\sigma = \frac{R^2 - R^1}{\sqrt{f_1 f_2}}\, dR^1 \wedge dR^2$$

is the area form of the metric $g^{11},\ g^{22}$. Thus, the magnetic field also makes an invariant sense. Notice that in the case when

$$f^1 = 4(R^1)^3 + a(R^1)^2 + bR^1 + c, \quad f^2 = -4(R^2)^3 - a(R^2)^2 - bR^2 - c$$

are cubic polynomials, the Gaussian curvature $K = 1$ and the operator H represents Dirac monopole on the unit sphere in the spherical-conical coordinates R^1, R^2. The scalar potential h (which in this case is proportional to $R^1 + R^2$) has a meaning of the external quadratic potential. We refer to [7] for the discussion of some algebraic aspects of spectral theory of such operators in the particular case $\epsilon_2 = 0$. The general situation will be discussed elsewhere.

The examples discussed in this section clearly demonstrate that there exists a one-to-one correspondence between commuting Schrödinger operators with magnetic fields and Lie-applicable surfaces which possess multi-parameter families of Lie deformations.

4 Canal surfaces

Our approach to the canal surfaces will be based on a linear system

$$\begin{aligned} \partial_1^2 \boldsymbol{\psi} &= -i\ p\ \partial_2 \boldsymbol{\psi} + \tfrac{1}{2}(V + i\ \partial_2 p)\ \boldsymbol{\psi} \\ \partial_2^2 \boldsymbol{\psi} &= \tfrac{1}{2} W\ \boldsymbol{\psi} \end{aligned} \tag{27}$$

which is a specialization of (5) corresponding to $q = 0$. The compatibility conditions of system (27) take the form

$$\begin{aligned} \partial_2^3 p - 2W \partial_2 p - p \partial_2 W = 0 \\ \partial_1 W = \partial_2 V = 0. \end{aligned} \tag{28}$$

Since $q = 0$, we cannot use formulae (12). Instead, we introduce the four vectors

$$\boldsymbol{\psi}, \quad \boldsymbol{\psi}_1 = \partial_1 \boldsymbol{\psi}, \quad \boldsymbol{\psi}_2 = \partial_2 \boldsymbol{\psi} - \frac{1}{2} \frac{\partial_2 p}{p} \boldsymbol{\psi}, \quad \boldsymbol{\eta} = \partial_1 \partial_2 \boldsymbol{\psi} - \frac{1}{2} \frac{\partial_2 p}{p} \partial_1 \boldsymbol{\psi} \tag{29}$$

which satisfy the linear system

$$\begin{aligned} \partial_1 \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} b & 0 & -ip & 0 \\ \frac{1}{2} k & 0 & 0 & 1 \\ -\frac{i}{2} pa & \frac{1}{2} k & \frac{1}{2} b & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} \\ \partial_2 \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} \frac{\partial_2 p}{p} & 0 & 1 & 0 \\ 0 & \frac{1}{2} \frac{\partial_2 p}{p} & 0 & 1 \\ \frac{1}{2} a & 0 & -\frac{1}{2} \frac{\partial_2 p}{p} & 0 \\ 0 & \frac{1}{2} a & 0 & -\frac{1}{2} \frac{\partial_2 p}{p} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\eta} \end{pmatrix} \end{aligned} \tag{30}$$

where the notation

$$k = -\partial_1 \partial_2 (\ln p), \quad a = W - \partial_2^2 (\ln p) - \frac{1}{2} (\partial_2 \ln p)^2, \quad b = V$$

is introduced. Compatibility conditions of equations (30) imply

$$\partial_1 \partial_2 \ln p = -k, \quad \partial_1 a = \partial_2 k + \frac{\partial_2 p}{p} k, \quad \partial_2 b = 0, \quad p\, \partial_2 a + 2\, a\, \partial_2 p = 0.$$

An important property of system (30) is the existence of the quadratic integral

$$-\boldsymbol{\psi} \bar{\boldsymbol{\eta}} + \boldsymbol{\psi}_1 \bar{\boldsymbol{\psi}}_2 + \boldsymbol{\psi}_2 \bar{\boldsymbol{\psi}}_1 - \boldsymbol{\eta} \bar{\boldsymbol{\psi}} \tag{31}$$

which defines an invariant pseudo-Hermitian scalar product of the signature $(2, 2)$ on the space of solutions of system (30). Using (29), this integral can be rewritten in the form

$$-\boldsymbol{\psi} \partial_1 \partial_2 \bar{\boldsymbol{\psi}} + \partial_1 \boldsymbol{\psi} \partial_2 \bar{\boldsymbol{\psi}} + \partial_2 \boldsymbol{\psi} \partial_1 \bar{\boldsymbol{\psi}} - \bar{\boldsymbol{\psi}} \partial_1 \partial_2 \boldsymbol{\psi}. \tag{32}$$

The invariant pseudo-Hermitian scalar product (31) implies the existence of a basis of solutions of (30) such that

$$(\boldsymbol{\psi}_1,\ \boldsymbol{\psi}_2) = (\boldsymbol{\psi}_2,\ \boldsymbol{\psi}_1) = 1, \qquad (\boldsymbol{\psi},\ \boldsymbol{\eta}) = (\boldsymbol{\eta},\ \boldsymbol{\psi}) = -1, \tag{33}$$

all other scalar products being zero. Equations (33) are obviously equivalent to

$$(\partial_1\boldsymbol{\psi}, \partial_2\boldsymbol{\psi}) = (\partial_2\boldsymbol{\psi}, \partial_1\boldsymbol{\psi}) = 1, \quad (\partial_1\partial_2\boldsymbol{\psi}, \boldsymbol{\psi}) = (\boldsymbol{\psi}, \partial_1\partial_2\boldsymbol{\psi}) = -1.$$

Here (,) denotes the pseudo-Hermitian scalar product in $\mathbf{C}^4$ of the signature (2, 2) as in Appendix C. Equations (30) also imply that the determinant $\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}$ is invariant:

$$\partial_1(\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}) = \partial_2(\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}) = 0.$$

Thus, besides (33), we can impose the additional constraint

$$\boldsymbol{\psi}\wedge\boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2\wedge\boldsymbol{\eta} = 1. \tag{34}$$

From now on, we fix a null-tetrad $\boldsymbol{\psi},\ \boldsymbol{\psi}_1,\ \boldsymbol{\psi}_2,\ \boldsymbol{\eta}$ satisfying both (33) and (34). Notice that such a basis is defined up to the action of the group $SU(2,2)$ which preserves both (33) and (34).

Introducing the basis in $\Lambda^2\mathbf{C}^4$ as follows

$$\mathcal{U} = i\ \boldsymbol{\psi}\wedge\boldsymbol{\psi}_1, \quad \mathcal{V} = \boldsymbol{\psi}\wedge\boldsymbol{\psi}_2,$$

$$\mathcal{A} = i\ \boldsymbol{\psi}_2\wedge\boldsymbol{\psi}_1 + i\ \boldsymbol{\psi}\wedge\boldsymbol{\eta}, \quad \mathcal{B} = \boldsymbol{\psi}_1\wedge\boldsymbol{\psi}_2 + \boldsymbol{\psi}\wedge\boldsymbol{\eta},$$

$$\mathcal{P} = 2i\,\boldsymbol{\psi}_2\wedge\boldsymbol{\eta}, \quad \mathcal{Q} = 2\,\boldsymbol{\psi}_1\wedge\boldsymbol{\eta},$$

we arrive at the equations

$$\partial_1\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & p & 0 & 0\\ k & 0 & 0 & 0 & 0 & 0\\ 0 & k & 0 & -pa & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & b & 0 & 1\\ pa & 0 & -p & 0 & b & 0\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}$$

$$\partial_2\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix} \frac{\partial_2 p}{p} & 1 & 0 & 0 & 0 & 0\\ a & 0 & 1 & 0 & 0 & 0\\ 0 & a & -\frac{\partial_2 p}{p} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}. \tag{35}$$

Let us define the pseudo-Hermitian scalar product (,) and the complex scalar product { , } in $\Lambda^2\mathbf{C}^4$ as in Appendix 3. A direct computation shows that the only nonzero products among the vectors $\mathcal{U}, \mathcal{A}, \mathcal{P}, \mathcal{V}, \mathcal{B}, \mathcal{Q}$ are

$$(\mathcal{U},\ \mathcal{P}) = (\mathcal{P},\ \mathcal{U}) = (\mathcal{V},\ \mathcal{Q}) = (\mathcal{Q},\ \mathcal{V}) = -1, \quad (\mathcal{A},\ \mathcal{A}) = (\mathcal{B},\ \mathcal{B}) = 1 \tag{36}$$

and

$$\{\mathcal{U},\ \mathcal{P}\} = \{\mathcal{P},\ \mathcal{U}\} = \{\mathcal{V},\ \mathcal{Q}\} = \{\mathcal{Q},\ \mathcal{V}\} = -1, \quad \{\mathcal{A},\ \mathcal{A}\} = \{\mathcal{B},\ \mathcal{B}\} = 1, \tag{37}$$

respectively. Thus,

$$(\xi,\ \psi) = \{\xi,\ \bar{\psi}\} \tag{38}$$

for any ξ, ψ in $\Lambda^2(\mathbf{C}^4)$. Introducing in $\Lambda^2(\mathbf{C}^4)$ the real vectors

$$\mathbf{U} = \mathcal{U} + \bar{\mathcal{U}} = i\ \boldsymbol{\psi} \wedge \partial_1 \boldsymbol{\psi} - i\ \overline{\boldsymbol{\psi} \wedge \partial_1 \boldsymbol{\psi}}, \qquad \mathbf{V} = \mathcal{V} + \bar{\mathcal{V}} = \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi} + \overline{\boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}},$$

we can formulate the main result of this section

Theorem 2 *Vectors* $\mathbf{U}$ *and* $\mathbf{V}$ *have zero norm:*

$$(\mathbf{U},\ \mathbf{U}) = (\mathbf{V},\ \mathbf{V}) = 0.$$

Moreover, the triple $\mathbf{U},\ \partial_2\mathbf{U},\ \partial_2^2\mathbf{U}$ *is orthogonal to the triple* $\mathbf{V},\ \partial_1\mathbf{V},\ \partial_1^2\mathbf{V}$. *Hence,* $\mathbf{U}$ *and* $\mathbf{V}$ *are curvature spheres of a surface. This surface will be a canal surface since* $\partial_2\mathbf{V} = 0$. *Any canal surface can be obtained (locally) by this construction.*

The proof of this theorem copies the proof of Theorem 1 from section 2: it readily follows from (35) that the triple $\mathbf{U},\ \partial_2\mathbf{U},\ \partial_2^2\mathbf{U}$ is equivalent to $\mathcal{U} + \bar{\mathcal{U}},\ \mathcal{V} + \bar{\mathcal{V}},\ \mathcal{P} + \bar{\mathcal{P}}$. Similarly, the triple $\mathbf{V},\ \partial_1\mathbf{V},\ \partial_1^2\mathbf{V}$ is equivalent to $\mathcal{V} + \bar{\mathcal{V}},\ \mathcal{B} + \bar{\mathcal{B}},\ \mathcal{Q} + \bar{\mathcal{Q}}$. The conditions $(\mathbf{U},\ \mathbf{U}) = (\mathbf{V},\ \mathbf{V}) = 0$ and the orthogonality of both triples follow by virtue of (36), (37) and (38).

Example. Let us consider the Landau operator

$$H = \frac{1}{2}(i\ \partial_x - My)^2 + \frac{1}{2}(i\ \partial_y)^2$$

describing a quantum particle in the homogeneous magnetic field ($M = \text{const}$). Operator H obviously commutes with the operator

$$F = -\partial_x^2$$

so that the equations for their joint eigenfunctions

$$H\boldsymbol{\psi} = \lambda\ \boldsymbol{\psi}, \qquad F\boldsymbol{\psi} = k^2\ \boldsymbol{\psi}$$

can be rewritten in the form

$$\begin{gathered} \boldsymbol{\psi}_{yy} = -2iMy\ \boldsymbol{\psi}_x + (M^2y^2 + k^2 - 2\lambda)\ \boldsymbol{\psi} \\ \boldsymbol{\psi}_{xx} = -k^2\ \boldsymbol{\psi}. \end{gathered} \tag{39}$$

System (39) is obviously of the form (27) under the identification $p = 2MR^1,\ V = 2M^2(R^1)^2 + 2k^2 - 4\lambda,\ W = -2k^2,\ y = R^1,\ x = R^2$. In what follows we assume $M = 1$

(this can be achieved by a rescaling $\tilde{x} = x\sqrt{M},\ \tilde{y} = y\sqrt{M},\ \tilde{k} = k/\sqrt{M},\ \tilde{\lambda} = \lambda/M$). The corresponding linear system for $\boldsymbol{\psi}$ possesses 4 linearly independent solutions

$$e^{ikx}\psi_1(y+k), \quad e^{-ikx}\psi_1(y-k), \quad e^{ikx}\psi_2(y+k), \quad e^{-ikx}\psi_2(y-k)$$

where $\psi_1,\ \psi_2$ form a basis of solutions of Hermite's equation

$$\psi'' = (y^2 - 2\lambda)\psi.$$

Let us introduce the complex 4-vector

$$\boldsymbol{\psi} = \left(e^{ikx}\psi_1(y+k),\ e^{-ikx}\psi_1(y-k),\ \frac{e^{-ikx}\psi_2(y-k)}{2ikW},\ \frac{e^{ikx}\psi_2(y+k)}{2ikW}\right)$$

where $W = \psi_1\psi_2' - \psi_2\psi_1' = \text{const}$ is the Wronskian. Then

$$(\boldsymbol{\psi}_x,\ \boldsymbol{\psi}_y) = (\boldsymbol{\psi}_y,\ \boldsymbol{\psi}_x) = 1, \quad (\boldsymbol{\psi},\ \boldsymbol{\psi}_{xy}) = (\boldsymbol{\psi}_{xy},\ \boldsymbol{\psi}) = -1,$$

all other products being zero. Moreover,

$$\boldsymbol{\psi} \wedge \boldsymbol{\psi}_x \wedge \boldsymbol{\psi}_y \wedge \boldsymbol{\psi}_{xy} = -1.$$

A direct computation implies

$$\mathbf{V} = \boldsymbol{\psi} \wedge \boldsymbol{\psi}_x + \overline{\boldsymbol{\psi} \wedge \boldsymbol{\psi}_x} = (y^0,\ y^1,\ y^2,\ y^3,\ y^4,\ y^5)$$

where

$$y^0 = \frac{\psi_1(y-k)\psi_2(y+k) - \psi_2(y-k)\psi_1(y+k)}{W},$$

$$y^1 = -\frac{\psi_1(y-k)\psi_2(y+k) + \psi_2(y-k)\psi_1(y+k)}{W},$$

$$y^2 = y^3 = 0,$$

$$y^4 = -2k\psi_1(y-k)\psi_1(y+k) + \frac{\psi_2(y-k)\psi_2(y+k)}{2kW^2},$$

$$y^5 = -2k\psi_1(y-k)\psi_1(y+k) - \frac{\psi_2(y-k)\psi_2(y+k)}{2kW^2}.$$

Obviously, $-(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 - (y^5)^2 = 0$. Dividing by $y^0 + y^1$, we obtain the normalised vector

$$\mathbf{V} = \boldsymbol{\psi} \wedge \boldsymbol{\psi}_x + \overline{\boldsymbol{\psi} \wedge \boldsymbol{\psi}_x} = \left(\frac{y^0}{y^0+y^1},\ \frac{y^1}{y^0+y^1},\ 0,\ 0,\ \frac{y^4}{y^0+y^1},\ \frac{y^5}{y^0+y^1}\right).$$

Since the centers of the corresponding spheres lie on the z-axis, our surface is a surface of revolution. Parametric equations of centers z and radii R are

$$z = \frac{y^4}{y^0+y^1} = kW\frac{\psi_1(y-k)}{\psi_2(y-k)} - \frac{1}{4kW}\frac{\psi_2(y+k)}{\psi_1(y+k)},$$

$$R = \frac{y^5}{y^0+y^1} = kW\frac{\psi_1(y-k)}{\psi_2(y-k)} + \frac{1}{4kW}\frac{\psi_2(y+k)}{\psi_1(y+k)}.$$

The Landau levels correspond to $\lambda = \frac{2n+1}{2}$. For $n = 0$ we have

$$\psi_1 = e^{-\frac{y^2}{2}}, \quad \psi_2 = e^{-\frac{y^2}{2}} \int_0^y e^{\xi^2}\, d\xi, \quad W = 1.$$

I would like to thank K.R. Khusnutdinova for the investigation of this and other examples. The details will be given elsewhere.

5 Appendix A: Wilczynski's projective frame

Based on [17] (see also [2], [10], [6], [5], [16]), let us briefly recall the standard way of defining surfaces M^2 in projective space P^3 in terms of solutions of a linear system (3) satisfying the compatibility conditions (4). For any fixed β, γ, V, W satisfying (4), the linear system (3) is compatible and possesses a solution $\mathbf{r} = (r^0, r^1, r^2, r^3)$ where $r^i(x, y)$ can be regarded as homogeneous coordinates of a surface in projective space P^3. In what follows, we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x and y are real. Even though the coefficients β, γ, V, W define a surface M^2 uniquely up to projective equivalence via (3), it is not entirely correct to regard β, γ, V, W as projective invariants. Indeed, the asymptotic coordinates x, y are only defined up to an arbitrary reparametrization of the form

$$x^* = f(x), \quad y^* = g(y) \tag{40}$$

which induces a scaling of the surface vector according to

$$\mathbf{r}^* = \sqrt{f'(x)g'(y)}\ \mathbf{r}. \tag{41}$$

Thus [2, p. 1], the form of equations (3) is preserved by the above transformation with the new coefficients $\beta^*, \gamma^*, V^*, W^*$ given by

$$\begin{aligned} \beta^* &= \beta g'/(f')^2, \qquad V^*(f')^2 = V + S(f) \\ \gamma^* &= \gamma f'/(g')^2, \qquad W^*(g')^2 = W + S(g), \end{aligned} \tag{42}$$

where $S(\,\cdot\,)$ is the Schwarzian derivative, that is

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The transformation formulae (42) imply that the symmetric 2-form

$$2\beta\gamma\, dxdy$$

and the conformal class of the cubic form

$$\beta\, dx^3 + \gamma\, dy^3$$

are absolute projective invariants. They are known as the projective metric and the Darboux cubic form, respectively, and play an important role in projective differential

geometry. The vanishing of the Darboux cubic form is characteristic for quadrics: indeed, in this case $\beta=\gamma=0$ so that asymptotic curves of both families are straight lines. The vanishing of the projective metric (which is equivalent to either $\beta=0$ or $\gamma=0$) characterises ruled surfaces. In what follows we exclude these two degenerate situations and require $\beta\neq 0$, $\gamma\neq 0$.

Using (40)-(55), one can verify that the four points

$$\begin{gathered}
\mathbf{r}, \quad \mathbf{r}_1=\mathbf{r}_x-\tfrac{1}{2}\tfrac{\gamma_x}{\gamma}\mathbf{r}, \quad \mathbf{r}_2=\mathbf{r}_y-\tfrac{1}{2}\tfrac{\beta_y}{\beta}\mathbf{r}, \\
\boldsymbol{\eta}=\mathbf{r}_{xy}-\tfrac{1}{2}\tfrac{\gamma_x}{\gamma}\mathbf{r}_y-\tfrac{1}{2}\tfrac{\beta_y}{\beta}\mathbf{r}_x+\left(\tfrac{1}{4}\tfrac{\beta_y\gamma_x}{\beta\gamma}-\tfrac{1}{2}\beta\gamma\right)\mathbf{r}
\end{gathered} \tag{43}$$

are defined in an invariant way, that is, under the transformation formulae (40)-(55) they acquire a nonzero multiple which does not change them as points in projective space P^3. These points form the vertices of the so-called Wilczynski moving tetrahedral [2], [8], [17]. Since the lines passing through $\mathbf{r},\mathbf{r}_1$ and $\mathbf{r},\mathbf{r}_2$ are tangential to the x- and y-asymptotic curves, respectively, the three points $\mathbf{r},\mathbf{r}_1,\mathbf{r}_2$ span the tangent plane of the surface M^2. The line through $\mathbf{r}_1,\mathbf{r}_2$ lying in the tangent plane is known as the directrix of Wilczynski of the second kind. The line through $\mathbf{r},\boldsymbol{\eta}$ is transversal to M^2 and is known as the directrix of Wilczynski of the first kind. It plays the role of a projective 'normal'. The Wilczynski tetrahedral proves to be the most convenient tool in projective differential geometry.

Using (3) and (43), we easily derive for $\mathbf{r},\mathbf{r}_1,\mathbf{r}_2,\boldsymbol{\eta}$ the linear equations [8, p. 42]

$$\begin{gathered}
\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix}_x=\begin{pmatrix}\frac{1}{2}\frac{\gamma_x}{\gamma} & 1 & 0 & 0\\ \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} & \beta & 0\\ \frac{1}{2}k & 0 & \frac{1}{2}\frac{\gamma_x}{\gamma} & 1\\ \frac{1}{2}\beta a & \frac{1}{2}k & \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma}\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix} \\
\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix}_y=\begin{pmatrix}\frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 & 0\\ \frac{1}{2}l & \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1\\ \frac{1}{2}a & \gamma & -\frac{1}{2}\frac{\beta_y}{\beta} & 0\\ \frac{1}{2}\gamma b & \frac{1}{2}a & \frac{1}{2}l & -\frac{1}{2}\frac{\beta_y}{\beta}\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix},
\end{gathered} \tag{44}$$

where we introduced the notation

$$\begin{gathered}
k=\beta\gamma-(\ln\beta)_{xy}, \quad l=\beta\gamma-(\ln\gamma)_{xy}, \\
a=W-(\ln\beta)_{yy}-\tfrac{1}{2}(\ln\beta)_y^2, \quad b=V-(\ln\gamma)_{xx}-\tfrac{1}{2}(\ln\gamma)_x^2.
\end{gathered} \tag{45}$$

The compatibility conditions of equations (44) imply

$$\begin{gathered}
(\ln\beta)_{xy}=\beta\gamma-k, \quad (\ln\gamma)_{xy}=\beta\gamma-l, \\
a_x=k_y+\tfrac{\beta_y}{\beta}k, \quad b_y=l_x+\tfrac{\gamma_x}{\gamma}l, \\
\beta a_y+2a\beta_y=\gamma b_x+2b\gamma_x,
\end{gathered} \tag{46}$$

which is just the equivalent form of the projective 'Gauss-Codazzi' equations (4).

Equations (44) can be rewritten in the Plücker coordinates. For a convenience of the reader we briefly recall this construction. Let us consider a line l in P^3 passing through the points $\mathbf{a}$ and $\mathbf{b}$ with the homogeneous coordinates $\mathbf{a} = (a^0 : a^1 : a^2 : a^3)$ and $\mathbf{b} = (b^0 : b^1 : b^2 : b^3)$. With the line l we associate a point $\mathbf{a} \wedge \mathbf{b}$ in projective space P^5 with the homogeneous coordinates

$$\mathbf{a} \wedge \mathbf{b} = (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}),$$

where

$$p_{ij} = \det \begin{pmatrix} a^i & a^j \\ b^i & b^j \end{pmatrix}.$$

The coordinates p_{ij} satisfy the well-known quadratic Plücker relation

$$p_{01}\, p_{23} + p_{02}\, p_{31} + p_{03}\, p_{12} = 0. \tag{47}$$

Instead of $\mathbf{a}$ and $\mathbf{b}$ we may consider an arbitrary linear combinations thereof without changing $\mathbf{a} \wedge \mathbf{b}$ as a point in P^5. Hence, we arrive at the well-defined Plücker coorrespondence $l(\mathbf{a}, \mathbf{b}) \to \mathbf{a} \wedge \mathbf{b}$ between lines in P^3 and points on the Plücker quadric in P^5. Plücker correspondence plays an important role in the projective differential geometry of surfaces and often sheds some new light on those properties of surfaces which are not 'visible' in P^3 but acquire a precise geometric meaning only in P^5. Thus, let us consider a surface $M^2 \in P^3$ with the Wilczynski tetrahedral $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\eta}$ satisfying equations (44). Since the two pairs of points $\mathbf{r}, \mathbf{r}_1$ and $\mathbf{r}, \mathbf{r}_2$ generate two lines in P^3 which are tangential to the x- and y-asymptotic curves, respectively, the formulae

$$\mathcal{U} = \mathbf{r} \wedge \mathbf{r}_1, \quad \mathcal{V} = \mathbf{r} \wedge \mathbf{r}_2$$

define the images of these lines under the Plücker embedding. Hence, with any surface $M^2 \in P^3$ there are canonically associated two surfaces $\mathcal{U}(x, y)$ and $\mathcal{V}(x, y)$ in P^5 lying on the Plücker quadric (47). In view of the formulae

$$\mathcal{U}_x = \beta\, \mathcal{V}, \quad \mathcal{V}_y = \gamma\, \mathcal{U},$$

we conclude that the line in P^5 passing through a pair of points $(\mathcal{U}, \mathcal{V})$ can also be generated by the pair of points $(\mathcal{U}, \mathcal{U}_x)$ (and hence is tangential to the x-coordinate line on the surface $\mathcal{U}$) or by a pair of points $(\mathcal{V}, \mathcal{V}_y)$ (and hence is tangential to the y-coordinate line on the surface $\mathcal{V}$). Consequently, the surfaces $\mathcal{U}$ and $\mathcal{V}$ are two focal surfaces of the congruence of straight lines $(\mathcal{U}, \mathcal{V})$ or, equivalently, $\mathcal{V}$ is the Laplace transform of $\mathcal{U}$ with respect to x and $\mathcal{U}$ is the Laplace transform of $\mathcal{V}$ with respect to y. We emphasize that the x- and y-coordinate lines on the surfaces $\mathcal{U}$ and $\mathcal{V}$ are not asymptotic but conjugate. Continuation of the Laplace sequence in both directions, that is taking the x-transform of $\mathcal{V}$, the y-transform of $\mathcal{U}$, etc., leads, in the generic case, to an infinite Laplace sequence in P^5 known as the Godeaux sequence of a surface M^2 [2, p. 344]. The surfaces of the Godeaux sequence carry important geometric information about the surface M^2 itself.

Introducing

$$\mathcal{A} = \mathbf{r}_2 \wedge \mathbf{r}_1 + \mathbf{r} \wedge \boldsymbol{\eta}, \quad \mathcal{B} = \mathbf{r}_1 \wedge \mathbf{r}_2 + \mathbf{r} \wedge \boldsymbol{\eta},$$

$$\mathcal{P} = 2\, \mathbf{r}_2 \wedge \boldsymbol{\eta}, \quad \mathcal{Q} = 2\, \mathbf{r}_1 \wedge \boldsymbol{\eta},$$

we arrive at the following equations for the Plücker coordinates:

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\beta a & 0 & \beta & 0 & b & -\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\gamma b & 0 & \gamma \\ \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}. \tag{48}$$

Equations (48) are consistent with the following table of scalar products:

$$(\mathcal{U}, \mathcal{P}) = -1, \quad (\mathcal{A}, \mathcal{A}) = 1, \quad (\mathcal{V}, \mathcal{Q}) = 1, \quad (\mathcal{B}, \mathcal{B}) = -1, \tag{49}$$

all other scalar products being equal to zero. This defines a scalar product of the signature (3, 3) which is the same as that of the quadratic form (47).

Different types of surfaces can be defined by imposing additional constraints on β, γ, V, W (respectively, $\beta, \gamma, k, l, a, b$), so that, in a sense, projective differential geometry is the theory of (integrable) reductions of the underdetermined system (4) (respectively, (46)). Although the three linear systems (3), (44) and (48) are in fact equivalent, some of them prove to be more suitable for studying particular classes of projective surfaces — see [6], [5] for the further discussion.

6 Appendix B: the Lie sphere frame

Here we describe the construction of the so-called Lie sphere frame canonically associated with a surface in Lie sphere geometry (see [5]). Although the construction follows essentially that of Blaschke [1], our final formulae prove to be more suitable for the purposes of this paper. Let M^2 be a surface in E^3 parametrized by coordinates R^1, R^2 of curvature lines, with the radius-vector $\mathbf{r}$ and the unit normal $\mathbf{n}$ satisfying the Weingarten equations (1). Introducing the 6-vectors

$$\mathbf{U} = \left\{ \frac{1 + \mathbf{r}^2 - 2w^1(\mathbf{r}, \mathbf{n})}{2}, \ \ \frac{1 - \mathbf{r}^2 + 2w^1(\mathbf{r}, \mathbf{n})}{2}, \ \ \mathbf{r} - w^1\mathbf{n}, \ \ w^1 \right\}$$

and

$$\mathbf{V} = \left\{ \frac{1 + \mathbf{r}^2 - 2w^2(\mathbf{r}, \mathbf{n})}{2}, \ \ \frac{1 - \mathbf{r}^2 + 2w^2(\mathbf{r}, \mathbf{n})}{2}, \ \ \mathbf{r} - w^2\mathbf{n}, \ \ w^2 \right\},$$

we readily verify that

$$(\mathbf{U}, \ \mathbf{U}) = (\mathbf{U}, \ \mathbf{V}) = (\mathbf{V}, \ \mathbf{V}) = 0 \tag{50}$$

where the scalar product of 6-vectors is defined by the indefinite quadratic form (2). In what follows we use the same notation (,) for both the scalar product defined by (2) as well as for the standard Euclidean scalar product in E^3; however, the dimension of vectors will clearly indicate which one has to be choosen.

A direct computation gives

$$
\begin{aligned}
\partial_1\mathbf{U} &= \partial_1 w^1\,\{-(\mathbf{r},\mathbf{n}),\ (\mathbf{r},\mathbf{n}),\ -\mathbf{n},\ 1\}\\
\partial_2\mathbf{U} &= \partial_2 w^1\,\{-(\mathbf{r},\mathbf{n}),\ (\mathbf{r},\mathbf{n}),\ -\mathbf{n},\ 1\} + \tfrac{w^2-w^1}{w^2}\,\{(\partial_2\mathbf{r},\mathbf{r}),\ -(\partial_2\mathbf{r},\mathbf{r}),\ \partial_2\mathbf{r},\ 0\}\\
\partial_1\mathbf{V} &= \partial_1 w^2\,\{-(\mathbf{r},\mathbf{n}),\ (\mathbf{r},\mathbf{n}),\ -\mathbf{n},\ 1\} + \tfrac{w^1-w^2}{w^1}\,\{(\partial_1\mathbf{r},\mathbf{r}),\ -(\partial_1\mathbf{r},\mathbf{r}),\ \partial_1\mathbf{r},\ 0\}\\
\partial_2\mathbf{V} &= \partial_2 w^2\,\{-(\mathbf{r},\mathbf{n}),\ (\mathbf{r},\mathbf{n}),\ -\mathbf{n},\ 1\}
\end{aligned}
\tag{51}
$$

implying

$$
\begin{aligned}
\partial_1\mathbf{U} &= \tfrac{\partial_1 w^1}{w^1-w^2}(\mathbf{U}-\mathbf{V})\\
\partial_2\mathbf{V} &= \tfrac{\partial_2 w^2}{w^2-w^1}(\mathbf{V}-\mathbf{U}).
\end{aligned}
\tag{52}
$$

Differentiating (50) and taking into account (51) and (52), we conclude that the only nonzero scalar products among the vectors $\mathbf{U},\ \mathbf{V},\ \partial_1\mathbf{U},\ \partial_2\mathbf{U},\ \partial_1\mathbf{V},\ \partial_2\mathbf{V}$ are the following:

$$
\begin{aligned}
(\partial_2\mathbf{U},\ \partial_2\mathbf{U}) &= (w^1-w^2)^2\,G_{22}\\
(\partial_1\mathbf{V},\ \partial_1\mathbf{V}) &= (w^1-w^2)^2\,G_{11}.
\end{aligned}
$$

Here $G_{11}=(\partial_1\mathbf{n},\partial_1\mathbf{n}),\ G_{22}=(\partial_2\mathbf{n},\partial_2\mathbf{n})$ are the components of the third fundamental form of the surface M^2. Differentiating the zero scalar products among $\mathbf{U},\ \mathbf{V},\ \partial_1\mathbf{U},\ \partial_2\mathbf{U},\ \partial_1\mathbf{V},\ \partial_2\mathbf{V}$ and keeping in mind (52), one can show that the triple $\mathbf{U},\partial_2\mathbf{U},\partial_2^2\mathbf{U}$ is orthogonal to the triple $\mathbf{V},\partial_1\mathbf{V},\partial_1^2\mathbf{V}$. In order to complete the vectors $\mathbf{U}$ and $\mathbf{V}$ to a frame in P^5 with the simplest possible table of scalar products, we will choose appropriate combinations among the triples $\mathbf{U},\partial_2\mathbf{U},\partial_2^2\mathbf{U}$ and $\mathbf{V},\partial_1\mathbf{V},\partial_1^2\mathbf{V}$, separately. Up to a certain normalization, the choice described below coincides with that from [1].

Let us introduce the normalized vectors

$$
\mathcal{U}=\frac{\mathbf{U}}{\sqrt{G_{22}}(w^2-w^1)},\qquad \mathcal{V}=\frac{\mathbf{V}}{\sqrt{G_{11}}(w^1-w^2)}. \tag{53}
$$

This normalization is convenient for several reasons: first of all, equations (52) reduce to the Dirac equation

$$
\begin{aligned}
\partial_1\mathcal{U} &= p\,\mathcal{V}\\
\partial_2\mathcal{V} &= q\,\mathcal{U}
\end{aligned}
\tag{54}
$$

with the coefficients p and q given by

$$
p=\frac{\partial_1 w^1}{w^1-w^2}\frac{\sqrt{G_{11}}}{\sqrt{G_{22}}},\qquad q=\frac{\partial_2 w^2}{w^2-w^1}\frac{\sqrt{G_{22}}}{\sqrt{G_{11}}}.
$$

It is important that both p are q are Lie-invariant (we emphasize that coefficients in (52) are not Lie-invariant). The reparametrization of coordinates

$$(R^1)^* = f(R^1), \quad (R^2)^* = g(R^2)$$

induces the transformation of p and q as follows:

$$p^* = pg'/(f')^2, \qquad q^* = qf'/(g')^2, \tag{55}$$

so that we can introduce the Lie-invariant metric

$$-pq\, dR^1 dR^2$$

and the Lie-invariant cubic form

$$p\,(dR^1)^3 - q\,(dR^2)^3 \tag{56}$$

(notice that only the conformal class of the cubic form does make an invariant sense).

There exist one more important property of the normalized vector $\mathcal{U}$ (resp., $\mathcal{V}$). It turns out that the action of the Lie sphere group in E^3 induces linear transformations of the coordinates of $\mathcal{U}$ (resp., $\mathcal{V}$). Since this linear action should necessarily preserve the Lie quadric (2), we arrive at the well-known isomorphism of the Lie sphere group and $SO(4,2)$. Thus, the normalization (53) linearises the action of the Lie sphere group (see [4] for the details).

The only nonzero scalar products among normalized vectors $\mathcal{U},\ \mathcal{V},\ \partial_1\mathcal{U},\ \partial_2\mathcal{U},\ \partial_1\mathcal{V},\ \partial_2\mathcal{V}$ are the following:

$$(\partial_2\mathcal{U},\ \partial_2\mathcal{U}) = (\partial_1\mathcal{V},\ \partial_1\mathcal{V}) = 1.$$

Obviously, the normalized triples $\mathcal{U}, \partial_2\mathcal{U}, \partial_2^2\mathcal{U}$ and $\mathcal{V}, \partial_1\mathcal{V}, \partial_1^2\mathcal{V}$ remain mutually orthogonal. Let us introduce the following vectors $\mathcal{A}, \mathcal{P}$ from the first triple:

$$\mathcal{A} = \partial_2\mathcal{U} - \frac{\partial_2 p}{p}\mathcal{U}, \qquad \mathcal{P} = \partial_2\mathcal{A} - a\,\mathcal{U}$$

which we require to have the following nonzero scalar products:

$$(\mathcal{A}, \mathcal{A}) = 1, \qquad (\mathcal{U}, \mathcal{P}) = -1.$$

This uniquely specifies

$$a = -\frac{1}{2}\,(\partial_2\mathcal{A}, \partial_2\mathcal{A}).$$

Similarly, we can choose the vectors

$$\mathcal{B} = \partial_1\mathcal{V} - \frac{\partial_1 q}{q}\,\mathcal{V}, \qquad \mathcal{Q} = \partial_1\mathcal{B} - b\,\mathcal{V}$$

with the nonzero scalar products

$$(\mathcal{B}, \mathcal{B}) = 1, \qquad (\mathcal{V}, \mathcal{Q}) = -1,$$

which fixes

$$b = -\frac{1}{2}\,(\partial_1\mathcal{B}, \partial_1\mathcal{B}).$$

Vectors $\mathcal{U},\ \mathcal{A},\ \mathcal{P}$ and $\mathcal{V},\ \mathcal{B},\ \mathcal{Q}$ constitute the Lie sphere frame with the following simple table of scalar products

$$(\mathcal{A},\mathcal{A}) = 1, \quad (\mathcal{U},\mathcal{P}) = -1, \quad (\mathcal{B},\mathcal{B}) = 1, \quad (\mathcal{V},\mathcal{Q}) = -1, \tag{57}$$

all other scalar products are zero, which is of the desired signature (4, 2).

Equations of motion of the Lie sphere frame can be conveniently represented in matrix form (19) (see [4])

$$\partial_1 \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & p & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -pa & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial_1 q}{q} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ pa & 0 & -p & 0 & b & -\frac{\partial_1 q}{q} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

$$\partial_2 \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix} = \begin{pmatrix} \frac{\partial_2 p}{p} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\frac{\partial_2 p}{p} & qb & 0 & -q \\ q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -qb & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}.$$

The compatibility conditions of (19) produce the equations (14)

$$\partial_1\partial_2 \ln p = pq - k, \qquad \partial_1\partial_2 \ln q = pq - l,$$

$$\partial_1 a = \partial_2 k + \tfrac{\partial_2 p}{p}\, k, \qquad \partial_2 b = \partial_1 l + \tfrac{\partial_1 q}{q}\, l,$$

$$p\,\partial_2 a + 2\,a\,\partial_2 p + q\,\partial_1 b + 2\,b\,\partial_1 q = 0,$$

which can be viewed as the "Gauss-Codazzi" equations in Lie sphere geometry. Another (equivalent) form of equations (14) can be obtained by introducing V and W

$$V = b + \partial_1^2 \ln q + \tfrac{1}{2}\,(\partial_1 \ln q)^2,$$

$$W = a + \partial_2^2 \ln p + \tfrac{1}{2}\,(\partial_2 \ln p)^2,$$

which, upon the substitution into (14), implies (6):

$$\partial_2^3 p - 2\,W\,\partial_2 p - p\,\partial_2 W + \partial_1^3 q - 2\,V\,\partial_1 q - q\,\partial_1 V = 0,$$

$$\partial_1 W = 2\,q\,\partial_2 p + p\,\partial_2 q$$

$$\partial_2 V = 2\,p\,\partial_1 q + q\,\partial_1 p.$$

7 Appendix C: the scalar products in $\Lambda^2(\mathbf{C}^4)$

Let us consider a space $\mathbf{C}^4$ equiped with the pseudo-Hermitian scalar product of the signature $(2,2)$

$$(\mathbf{a},\mathbf{b}) = -a^0\bar{b}^3 + a^1\bar{b}^2 + a^2\bar{b}^1 - a^3\bar{b}^0,$$

and define the wedge product $\mathbf{a}\wedge\mathbf{b} \in \Lambda^2(\mathbf{C}^4)$ of the vectors $\mathbf{a} = (a^0, a^1, a^2, a^3)$ and $\mathbf{b} = (b^0, b^1, b^2, b^3)$ by the formula

$$\mathbf{a}\wedge\mathbf{b} = (y^0, y^1, y^2, y^3, y^4, y^5)$$

where

$$y^0 = \tfrac{1}{2}(p_{02} - p_{31}), \quad y^1 = \tfrac{1}{2}(p_{02} + p_{31}), \quad y^2 = \tfrac{1}{2}(p_{03} + p_{12}),$$

$$y^3 = \tfrac{1}{2i}(p_{03} - p_{12}), \quad y^4 = \tfrac{1}{2i}(p_{01} - p_{23}), \quad y^5 = \tfrac{1}{2i}(p_{01} + p_{23}).$$

($p_{ij} = a^ib^j - a^jb^i$). Let

$$\mathbf{A}\wedge\mathbf{B} = (Y^0, Y^1, Y^2, Y^3, Y^4, Y^5)$$

be the wedge product of any other two vectors $\mathbf{A} = (A^0, A^1, A^2, A^3)$ and $\mathbf{B} = (B^0, B^1, B^2, B^3)$. The pseudo-Hermitian scalar product (,) in $\mathbf{C}^4$ induces the pseudo-Hermitian scalar product (,) in $\Lambda^2(\mathbf{C}^4)$ as follows:

$$(\mathbf{a}\wedge\mathbf{b},\ \mathbf{A}\wedge\mathbf{B}) = \frac{1}{2}((\mathbf{a},\mathbf{B})(\mathbf{b},\mathbf{A}) - (\mathbf{a},\mathbf{A})(\mathbf{b},\mathbf{B})).$$

In terms of y^i and Y^i this pseudo-Hermitian scalar product takes the form

$$(\mathbf{a}\wedge\mathbf{b},\ \mathbf{A}\wedge\mathbf{B}) = -y^0\bar{Y}^0 + y^1\bar{Y}^1 + y^2\bar{Y}^2 + y^3\bar{Y}^3 + y^4\bar{Y}^4 - y^5\bar{Y}^5,$$

which is of the signature $(4,2)$. Let us also define the complex scalar product $\{\ ,\ \}$ in $\Lambda^2(\mathbf{C}^4)$

$$\{\mathbf{a}\wedge\mathbf{b},\ \mathbf{A}\wedge\mathbf{B}\} = \frac{1}{2}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{A}\wedge\mathbf{B})$$

(here the right hand side is undestood as a half of the determinant of the 4×4 matrix with the rows $\mathbf{a},\ \mathbf{b},\ \mathbf{A},\ \mathbf{B}$). In terms of y^i and Y^i this scalar product takes the form

$$\{\mathbf{a}\wedge\mathbf{b},\ \mathbf{A}\wedge\mathbf{B}\} = -y^0Y^0 + y^1Y^1 + y^2Y^2 + y^3Y^3 + y^4Y^4 - y^5Y^5.$$

Clearly,

$$(\xi,\ \psi) = \{\xi,\ \bar{\psi}\}$$

for any ξ, ψ in $\Lambda^2(\mathbf{C}^4)$.

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